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The Hahn–Banach theorem almost everywhere

Roman Badora

Dedicated to Professor Roman Ger on the occasion of his seventieth birthday

Abstract. The aim of this work is to present an almost everywhere version of the Hahn–Banach extension theorem.

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1. Introduction

In the year 1960 Erdös [3] raised the following problem: suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(x + y) = f(x) + f(y),$$

for almost all $(x, y) \in \mathbb{R}^2$ (in the sense of the planar Lebesque measure). Does there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ [i.e. a satisfies $a(x + y) = a(x) + a(y)$, for all $(x, y) \in \mathbb{R}^2$] such that

$$f(x) = a(x)$$

almost everywhere in $\mathbb{R}$ (in the sense of the linear Lebesque measure)? A positive answer to this question was given by de Bruijn [2] (and, independently, by Jurkat [9]). N. G. de Bruijn has put the Erdös problem into a more general setting.

Let $(G, +)$ be a group. A non-empty family $\mathcal{I}$ of subsets of G is called a proper linearly invariant ideal (briefly p.l.i. ideal) iff it satisfies the following conditions

(i) $G \notin \mathcal{I}$;
(ii) if $U \in \mathcal{I}$ and $V \subset U$, then $V \in \mathcal{I}$;
(iii) if $U, V \in \mathcal{I}$, then $U \cup V \in \mathcal{I}$;
(iv) if $U \in \mathcal{I}$ and $x \in G$, then $x - U \in \mathcal{I}$.

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For a p.l.i. ideal $\mathcal{I}$ of subsets of a group G we say that a given condition is satisfied $\mathcal{I}$-*almost everywhere* in G (written $\mathcal{I}$-*a.e.*) iff there exists a set $Z \in \mathcal{I}$ such that this condition is satisfied for every $x \in G \backslash Z$.

The set belonging to the set ideal are regarded as, in a certain sense, small sets (see Kuczma [10]). For example, if G is a second category topological commutative group then the family of all first category subsets of G is a p.l.i. ideal. If G is a commutative locally compact topological group equipped with the Haar measure μ then the family of all subsets of G which have zero measure is a p.l.i. ideal. Moreover, if G is a normed space ($\dim G \geq 1$) then the family of all bounded subsets of G is a p.l.i. ideal and also, if G is a commutative uncountable group then the family of all countable subsets of G is a p.l.i. ideal.

Let $(G, +)$ be a commutative group. For a p.l.i. ideal $\mathcal{I}$ we may define the following family of subsets of $G \times G$ (Ger [5,6]):

$$\Omega(\mathcal{I}) = \{N \subset G \times G : N[x] \in \mathcal{I} \ \ \mathcal{I}\text{-}a.e. \ \text{ in } \ G\},$$

where

$$N[x] = \{y \in G : (x, y) \in N\}$$

a subset N of $G \times G$ belongs to $\Omega(\mathcal{I})$ iff there exists a set $U \in \mathcal{I}$ such that

$$N[x] \in \mathcal{I}, \ \ x \in G \backslash U$$

(an abstract version of the Fubini theorem)]. The family $\Omega(\mathcal{I})$ is a p.l.i. ideal of subsets of $G \times G$.

The de Bruijn result can be formulated as follows:

Theorem 1.1. *If $(G, +)$ and $(H, +)$ are commutative groups, $\mathcal{I}$ is a p.l.i. ideal of subsets of G then for every $\Omega(\mathcal{I})$-almost additive function $f : G \to H$, i.e.*

$$f(x + y) = f(x) + f(y) \quad \Omega(\mathcal{I})\text{-}a.e. \quad \text{in} \quad G \times G,$$

there exists a unique homomorphism $a : G \to H$ such that

$$f(x) = a(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad G.$$

Ger [7] generalized de Bruijn's theorem to the case of non-commutative groups. The notion of p.l.i. ideals and its properties and applications we can find in [10]. One of the most interesting applications is included in the paper of Ger [8] where the author combines the notions of approximately additive and almost additive mappings.

In this paper we proved the following version of the Hahn–Banach extension theorem.

Theorem 1.2. *Let $(H, +)$ be a subgroup of a commutative group $(G, +)$, let $\mathcal{I}$ be a p.l.i. ideal of subsets of G and let $H \notin \mathcal{I}$. Assume that $p : G \to \mathbb{R}$ satisfies*

$$p(x + y) \leq p(x) + p(y) \quad \Omega(\mathcal{I})\text{-}a.e. \quad \text{in} \quad G \times G. \tag{1.1}$$

Then for every additive function $a : H \to \mathbb{R}$ *fulfilling*

$$a(x) \le p(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad H \tag{1.2}$$

there exists an additive function $A : G \to \mathbb{R}$ *such that*

$$A(x) = a(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad H. \tag{1.3}$$

and

$$A(x) \le p(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad G. \tag{1.4}$$

2. Proof of the theorem

Assume that $\mathcal{I}$ is a p.l.i. ideal of subsets of a commutative group $(G, +)$. For a real function f on G we define $\mathcal{I}_f$ to be the family of all sets $Z \in \mathcal{I}$ such that f is bounded on the complement of Z. A real function f on G is called $\mathcal{I}$-*essentially bounded* if and only if the family $\mathcal{I}_f$ is non-empty. The space of all $\mathcal{I}$-essentially bounded functions on G will be denoted by $B^{\mathcal{I}}(R, \mathbb{R})$.

For every element f of the space $B^{\mathcal{I}}(G, \mathbb{R})$ the real numbers

$$\mathcal{I}\text{-essinf}_{x\in G} f(x) = \sup_{Z\in\mathcal{I}_f} \inf_{x\in G\setminus Z} f(x),$$
$$\mathcal{I}\text{-esssup}_{x\in G} f(x) = \inf_{Z\in\mathcal{I}_f} \sup_{x\in G\setminus Z} f(x)$$

are correctly defined and are referred to as the $\mathcal{I}$-*essential infimum* and the $\mathcal{I}$-*essential supremum* of the function f, respectively.

From the Gajda theorem (Gajda [4], see also Badora [1]) we can derive the following.

Theorem 2.1. *If* $\mathcal{I}$ *is a p.l.i. ideal of subsets of a commutative group* $(G, +)$, *then there exists a real linear functional* $M^{\mathcal{I}}$ *on the space* $B^{\mathcal{I}}(G, \mathbb{R})$ *such that*

$$\mathcal{I}\text{-}essinf_{x\in G} f(x) \le M^{\mathcal{I}}(f) \le \mathcal{I}\text{-}esssup_{x\in G} f(x) \tag{2.1}$$

and

$$M^{\mathcal{I}}({}_z f) = M^{\mathcal{I}}(f), \tag{2.2}$$

for all $f \in B^{\mathcal{I}}(G, \mathbb{R})$ *and all* $z \in G$, *where the function* ${}_z f : G \to \mathbb{R}$ *is defined as follows*

$${}_z f(x) = f(z + x), \quad x \in G.$$

Now we prove our result.

Proof of Theorem 1.2. Notice that if $\mathcal{I}$ is a p.l.i. ideal of subsets of G, then the family

$$\mathcal{I} \cap H = \{Z \cap H : Z \in \mathcal{I}\}$$

is a p.l.i. ideal of subsets of H.

From condition (1.1) we infer the existence of the set $U_1 \in \mathcal{I}$ such that for every $x \in G\backslash U_1$ there exists a set $V_x \in \mathcal{I}$ such that

$$p(x+y) \leq p(x) + p(y), \quad g \in G\backslash V_x. \tag{2.3}$$

From (1.2) it follows that there exists a set $U_0 \in \mathcal{I}$ such that

$$a(x) \leq p(x), \quad x \in H\backslash U_0. \tag{2.4}$$

Let $U = U_1 \cup (-U_1)$. Next we choose arbitrary $x \in G\backslash U$. By (2.4) and (2.3) we get

$$a(z) \leq p(z) = p(x - x + z) \leq p(x) + p(-x+z), \tag{2.5}$$

for all $z \in H\backslash(U \cup U_0 \cup (x + V_x))$. From (iv) with $x = 0$ we get $-V_x \in \mathcal{I}$. Using again (iv) we infer that $x + V_x \in \mathcal{I}$. Moreover $U \in \mathcal{I}$. Therefore (2.5) means that the function

$$H \ni z \mapsto a(z) - p(-x+z) \in \mathbb{R}$$

is $\mathcal{I}$-essentially bounded from above. So, we can define the function $\varphi : G \to \mathbb{R}$ as follows

$$\varphi(x) = \begin{cases} \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-x+z)), & x \in G\backslash U \\ 0, & x \in U. \end{cases}$$

Let $N = (U \times G) \cup (G \times U) \cup \{(x,y) \in G \times G : x + y \in U\}$. For every $x \in G\backslash U$ we have

$$\begin{aligned} N[x] &= \emptyset \cup U \cup \{y \in G : x + y \in U\} \\ &= U \cup \{y \in G : y \in -x + U\} = U \cup (-x + U) \in \mathcal{I}. \end{aligned}$$

Consequently $N \in \Omega(\mathcal{I})$.

Let $(x,y) \in G \times G\backslash N$ be fixed. Then $x \notin U$, $y \notin U$ and $x + y \notin U$. For $z \in G\backslash((y + V_{-x}) \cup (x + y + V_x))$, by (2.3), we have

$$p(-x-y+z) \leq p(-x) + p(-y+z)$$

and

$$p(-y+z) = p(x - x - y + z) \leq p(x) + p(-x-y+z)$$

which leads to the following

$$-p(-x) \leq p(-y+z) - p(-x-y+z) \leq p(x).$$

From this we get

$$\begin{aligned} \varphi(x+y) &= \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-y-x+z)) \\ &= \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-y+z) + p(-y+z) - p(-x-y+z)) \\ &\leq \varphi(y) + p(x) \end{aligned}$$

and

$$\begin{aligned}\varphi(y) &= \mathcal{I}\text{-esssup}_{z\in H}(a(z)-p(-y+z))\\ &= \mathcal{I}\text{-esssup}_{z\in H}(a(z)-p(-x-y+z)+p(-x-y+z)-p(-y+z))\\ &\le \varphi(x+y)+p(-x).\end{aligned}$$

Hence

$$-p(-x)\le \varphi(x+y)-\varphi(y)\le p(x),\quad (x,y)\in G\times G\backslash N. \tag{2.6}$$

The last inequalities imply that, for $x\in G\backslash U$, the function

$$G\backslash(U\cup(-x+U))\ni y\mapsto \varphi(x+y)-\varphi(y)\in\mathbb{R}$$

is bounded which yields that the function

$$G\ni y\mapsto \varphi(x+y)-\varphi(y)\in\mathbb{R}$$

belongs to the space $B^{\mathcal{I}}(G,\mathbb{R})$.

A function $\alpha : G\to\mathbb{R}$ we define by the formula

$$\alpha(x)=\begin{cases} M_y^{\mathcal{I}}(\varphi(x+y)-\varphi(y)), & x\in G\backslash U\\ 0, & x\in U,\end{cases}$$

where $M^{\mathcal{I}}$ is a linear functional whose existence guarantees Theorem 2.1 and the subscript y indicates that the functional $M^{\mathcal{I}}$ is applied to a function of the variable y.

If we choose $(x,y)\in G\times G\backslash N$ then, by the linearity of $M^{\mathcal{I}}$ and (2.2), we get

$$\begin{aligned}\alpha(x)+\alpha(y) &= M_z^{\mathcal{I}}(\varphi(x+z)-\varphi(z))+M_z^{\mathcal{I}}(\varphi(y+z)-\varphi(z))\\ &= M_z^{\mathcal{I}}(\varphi(x+y+z)-\varphi(y+z))+M_z^{\mathcal{I}}(\varphi(y+z)-\varphi(z))\\ &= M_z^{\mathcal{I}}(\varphi(x+y+z)-\varphi(z))=\alpha(x+y).\end{aligned}$$

The function α is $\Omega(\mathcal{I})$-almost additive and from Theorem 1.1 we obtain the existence of an additive function $A : G\to\mathbb{R}$ such that

$$A(x)=\alpha(x)\quad \mathcal{I}\text{-}a.e.\quad \text{in}\quad G. \tag{2.7}$$

Next, let $x\in H\backslash U$ be fixed and let $y\in G\backslash(U\cup(-x+U)\cup N[x])$. Then

$$\begin{aligned}\varphi(x+y) &= \mathcal{I}\text{-esssup}_{z\in H}(a(z)-p(-y-x+z))\\ &= \inf_{Z\in\mathcal{I}}\sup_{z\in H\backslash Z}(a(z)-p(-y-x+z))\\ &= \inf_{Z\in\mathcal{I}}\sup_{z\in H\backslash(-x+Z)}(a(x+z)-p(-y+z))\\ &= \inf_{Z\in\mathcal{I}}\sup_{z\in H\backslash(-x+Z)}(a(x)+a(z)-p(-y+z))\\ &= a(x)+\varphi(y).\end{aligned}$$

Therefore, for $x\in H\backslash U$, using (2.1) we have

$$\alpha(x)=M_y^{\mathcal{I}}(\varphi(x+y)-\varphi(y))=M_y^{\mathcal{I}}(a(x))=a(x),$$

which jointly with (2.7) gives (1.3) . Finally, from the definition of α, (2.1), (2.6) and (2.7) we infer that condition (1.4) is satisfied which ends the proof. $\square$

3. Ending comments

Remark 3.1. Note that we can strengthen Theorem 1.2 assuming that the function a is $\mathcal{I}$-almost additive. Then we start the proof from Theorem 1.1.

Remark 3.2. If, in Theorem 1.2, additionally we assume that the functional p is positively homogeneous then we can prove that the function A is linear.

Indeed, for a fixed $x \in G$ let us observe that condition (1.4) implies that

$$A(tx) \leq p(tx) = tp(x), \quad t \in (0, +\infty),$$

which means that the real additive function

$$\mathbb{R} \ni t \mapsto A(tx) \in \mathbb{R}$$

is bounded from above, for example, on the interval $(0, 1)$. Whence this function is linear (continuous), i. e.

$$A(tx) = t \cdot c_x, \quad t \in \mathbb{R},$$

for some constant $c_x \in \mathbb{R}$. Putting $t = 1$ we get $c_x = A(x)$ and

$$A(tx) = tA(x), \quad t \in \mathbb{R}.$$

Hence A is a linear map.

Remark 3.3. We will show that the assumption (iv) imposed on the family $\mathcal{I}$ (symmetry with respect to zero) is essential in our theorem.

Let $G = \mathbb{R}$ and let $H = \mathbb{Z}$. The family $\mathcal{I}$ of subsets of $\mathbb{R}$ we define as follows: $A \in \mathcal{I}$ iff A is a countable subset of the interval $(c, +\infty)$, for some $c \in \mathbb{R}$.

Then $H \notin \mathcal{I}$, the family $\mathcal{I}$ satisfies conditions (i)–(iii) of the definition of a p.l.i. ideal [but the condition (iv) is not fulfilled].

Taking $p : \mathbb{R} \to \mathbb{R}$ as $p(x) = \mid x \mid$, for $x \in \mathbb{R}$, and $a : \mathbb{Z} \to \mathbb{Z}$ as $a(x) = 2x$, for $x \in \mathbb{Z}$, we have that p satisfies (1.1) and a, p fulfill (1.2) [because $\mathbb{Z} \cap (0, +\infty) \in \mathcal{I}$].

If $A : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying (1.4), then it is bounded from above on some interval (p is bounded from above on each bounded interval). Therefore A is a linear map. So, $A(x) = cx$, $x \in \mathbb{R}$, for some constant $c \in \mathbb{R}$. Moreover $\mathbb{Z} \notin \mathcal{I}$ and if the function A fulfills (1.3), then $A(x) = 2x$, for $x \in \mathbb{R}$, which is impossible because A, p satisfy (1.4) and $(0, +\infty) \notin \mathcal{I}$.

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Roman Badora
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: robadora@math.us.edu.pl

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